SENRA Academic Publishers, Burnaby, British Columbia
Vol. 3. No. 2, pp. 827-832, 2009
ISSN: 1715-9997

Canadian Journal of pure&applied sciences

AN ANALYTICAL MODEL FOR RAIN-DROP SIZE DISTRIBUTION

*S S De[1], P Das[1], A Debnath, B Bandyopadhyay[1], S K Sarkar[2] and Suman Paul[1]
[1]Centre of advanced Study in Radio Physics and Electronics, 1, Girish Vidyaratna Lane, Kolkata 700 009
[2]Radio Science Division, National Physical Laboratory, New Delhi 110 012, India

ABSTRACT

An empirical model of raindrop size distribution has been formulated in this paper through which the relationship between integral parameters with the rainfall rate is deduced through a general form of gamma distribution function. It suitably connects the radar reflectivity factor, attenuation and other integral parameters with the rainfall rate. The variation of parameters in the distribution function has been worked out empirically. Numerical analyses of the fluctuations of integral parameters and distribution parameters have been carried out which are compared with some earlier works.

Keywords: Raindrop size distribution, radar reflectivity-rain rate relationship.

INTRODUCTION

Radar reflectivity factor and rainfall rate relation plays an important role in weather radar measurements, which specially depends on drop concentration and mean drop size, i.e., the type of rainfall. Earlier investigators in this field used different formulations for raindrop size distribution with various parameters along with different local characteristics to fit the observed spectra. Introducing two-parameter exponential distribution function, Marshall and Palmer (1948) deduced the connectivity between the reciprocal of the mean diameter and the rainfall rate. But, the instant shape of the raindrop size distribution what they obtained, usually differed from the predicted exponential form. Later, three-parameter gamma distribution function had been introduced by Ulbrich and Atlas (1975) to relate the rainfall rate with other integral parameters. Their investigations show higher rainfall rates above the actual value obtained by X-band radar. Considering radar reflectivity factor (*Z*) and optical extinction (Σ) as dual parameters, Atlas and Ulbrich (1977) obtained some deviations in the estimation of rainfall-rate below the actual value. They introduced shape-factor in the distribution function for analysis to match the experimental results. Later, a linear relationship between the radar reflectivity factor and rainfall rate was presented by List (1988). But it showed deviations from earlier results. To get better correlation, Ulbrich (1992), Torres *et al.* (1994) proposed a power relationship between the integral parameters which was a good fit with the experimental data.

Thus, different attempts have been made to build up appropriate model connecting different parameters for satisfactory explanation of the observed raindrop size distribution (Ulbrich and Atlas, 1998; Timothy *et al.*, 2002; Asen and Gibbins, 2002).

In this paper, an empirical model has been formulated to explore the situation. The gamma distribution function for rain-drop is chosen suitably through which the expressions of the characteristics of radar reflectivity factor (*Z*) and rainfall rate (*R*) have been deduced. The numerical computations are carried out to study the variation of the basic parameters. The results are presented graphically along with an earlier work (Feingold and Levin, 1986).

MATHEMATICAL FORMULATIONS

In the present model, the form of the three-parameter gamma distribution functions has been taken as

$$N(D) = N_0 \frac{D^{(\mu-1)} \exp(-\Lambda D)}{\Gamma(\mu)} \qquad (1)$$

where, *D* is the drop diameter and *N*(*D*) the number density. N_0, μ and Λ are three parameters representing concentration scaling parameter, distribution shape factor and slope coefficient, respectively. The general form of three-parameter gamma distribution function is chosen which is capable of describing a broader variation in raindrop size distribution than an exponential distribution. One of the integral parameters of interest is the rainfall rate *R*, defined by

$$R = \frac{6\pi}{10^4} \int_0^\infty N(D) D^3 v(D) dD \qquad (2)$$

where, v (*D*) is the terminal velocity of rain-drop in still

*Corresponding author email: de_syam_sundar@yahoo.co.in

air, and at standard pressure and temperature. In equation (2), it has been assumed that there are no effects of turbulence and vortex on $v(D)$. The effects of wind are also neglected and all the calculations are done in the ground level. The terminal velocity has been taken from Atlas and Ulbrich (1977). In the presence of turbulence, for high Reynolds number, very often the modified velocity expression is taken into account (Best, 1950; Beard, 1976). The widely used form of $v(D)$ is the power law, given by

$$v(D) = cD^{\gamma} \tag{3}$$

where, D varies from 0.5 mm to 5 mm. This occurs for the most type of rainfalls with the coefficients $\gamma = 0.67$ and $c = 17.67$ m.s^{-1}mm$^{-0.67}$, (Atlas and Ulbrich, 1977).

From equations (3) and (2), the expression of rain rate (R) can be derived as

$$R = \frac{6\pi}{10^4}\int_0^{\infty} N_0 \frac{D^{(\mu-1)}\exp(-\Lambda D)}{\Gamma(\mu)} D^3 cD^{\gamma}\ dD$$

On integration,

$$R = \frac{6\pi N_0 c\Gamma(3+\gamma+\mu)}{10^4\Gamma(\mu)\Lambda^{(3+\gamma+\mu)}} \tag{4}$$

$\Gamma(x)$ is the complete gamma function, i.e.,

$$\Gamma(x) = \int_0^{\infty} u^{x-1}\exp(-u)dx \tag{5}$$

Another integral parameter is the radar reflectivity factor Z, defined as

$$Z = \int_0^{\infty} N(D)D^6\ dD \tag{6}$$

Integrating (6) with the use of (1), one can get

$$Z = \frac{N_0}{\Lambda^{6+\mu}}\frac{\Gamma(6+\mu)}{\Gamma(\mu)} \tag{7}$$

It is found that all the dual measurement methods for this work involve the 'measurable (P)' which can be represented by

$$P = c_P \int_{D_{\min}}^{D_{\max}} D^P N(D)\ dD \tag{8}$$

where, c_p and p are constants and varies with the 'measurable'. D_{min} and D_{max} are minimum and maximum diameters, respectively. Here it has been assumed that $D_{min} \to 0$ which is physically reasonable. But $D_{max} \to \infty$ has been chosen just for computation (Ulbrich, 1985). For all 'measurables', a suitably produced table is known (Ulbrich, 1992). Choosing the total number of drops N_T per unit volume of air to be

$$N_T = \int_0^{\infty} N(D)\ dD = \frac{N_0}{\Lambda^{\mu}} \tag{9}$$

the mass weighed mean diameter D_m as (Ulbrich and Atlas, 1998).

$$D_m = \frac{p_4}{p_3} \tag{10}$$

with p_4 and p_3 as the 4th and the 3rd moments of DSD respectively, and the n^{th} moment

$$p_n = \int D^n N(D)\ dD\,, \tag{11}$$

one can get from equations (10) and (11),

$$D_m = \frac{1}{\Lambda}\left[\frac{\Gamma(4+\mu)}{\Gamma(3+\mu)}\right] = \frac{3+\mu}{\Lambda} \tag{12}$$

Following Ulbrich and Atlas (1998), the median volume diameter (D_0), can be written as $D_0 = \frac{3.67+\mu}{\Lambda}$, which takes the form

$$D_0 = \frac{3.67+\mu}{3+\mu}D_m \tag{13}$$

From equation (9) and (7), one can deduce

$$Z = N_T\left[\frac{D_m}{3+\mu}\right]^6 \frac{\Gamma(6+\mu)}{\Gamma(\mu)} \tag{14}$$

Thus, the rain rate equation (4) becomes

$$R = \frac{6\pi c}{10^4}\frac{N_T}{\Lambda^{3+\gamma}}\frac{\Gamma(3+\gamma+\mu)}{\Gamma(\mu)}$$

$$= \frac{6\pi c}{10^4}N_T\left[\frac{D_m}{3+\mu}\right]^{3+\gamma}\frac{\Gamma(3+\gamma+\mu)}{\Gamma(\mu)} \tag{15}$$

Equations (14) and (15) signify that Z and R are directly proportional to the total number of drops per unit volume, whereas Z is proportional to the 6th power of the mean diameter and R is proportional to the $(3+\gamma)^{th}$ power of the mean diameter. It is notable that prefactors of these expressions are solely dependent on μ (shape parameter of the distribution function).

Z-R relations

Generally, the relationship between the radar reflectivity factor (Z) and rain rate (R) is not unique. Actually, there is a fundamental uncertainty in the Z-R relationship. However, it is possible to derive Z-R relationship on the basis of raindrop size distribution and the terminal velocity v (D) of the drops at the ground level. Empirically, it can be shown that there exists a Z-R relationship in the form of power law. The relationship can be derived by substituting the total number of drops per unit volume (N_T) in equation (15), which yields

$$Z = \frac{10^4}{6\pi c}\left[\frac{D_m}{3+\mu}\right]^{3-\gamma}\frac{\Gamma(6+\mu)}{\Gamma(3+\gamma+\mu)}R \tag{16}$$

The expression (16) is analogous to the form $Z=C_zR$ (List, 1988) for equilibrium rainfall condition which are observed during steady tropical rain. Here the prefactor depends on the mean diameter (D_m) and the distribution shape parameter (μ). For the case of constant D_m and valid range of μ, the prefactor in equation (16) may be determined. The above relation remains valid even if D_m is not constant. In that case, D_m–R relationship is required such that Z–R relationship may be adjusted.

From equation (15) and (16) one can get

$$Z=\left[\frac{10^4}{6\pi c}\frac{1}{\Gamma(3+\gamma+\mu)}\right]^{\frac{6}{3+\gamma}} N_T^{\frac{\gamma-3}{\gamma+3}}\left(R\Gamma(\mu)\right)^{\frac{3-\gamma}{3+\gamma}}\Gamma(6+\mu) \qquad (17)$$

Equation (17) can be treated as a power relationship between Z–R, where the number of drops per unit volume (N_T) is occurring in the prefactor. This prefactor may be determined from N_T–R relationship.

Combining equation (9), (12) and (4), another Z–R relationship can be deduced, which is given by

$$Z=\left[\frac{10^4}{6\pi c}\right]^{\frac{(6+\mu)}{(3+\gamma+\mu)}}\left[\frac{\Gamma(\mu)}{\Gamma(3+\gamma+\mu)}\right]^{\frac{(6+\mu)}{(3+\gamma+\mu)}}\frac{\Gamma(6+\mu)}{\Gamma(\mu)}\times$$

$$\times\left[N_T\left(\frac{3+\mu}{D_m}\right)^{\mu}\right]^{\frac{\gamma-3}{3+\gamma+\mu}}\left(\frac{3+\mu}{D_m}\right)^{-\frac{3\mu}{3+\gamma+\mu}} R^{\frac{6+\mu}{3+\gamma+\mu}} \qquad (18)$$

The Z-R relationship is alike to the general form obtained from *Torres et al.* (1994) given by

$$Z=aR^b \qquad (19)$$

where, a is the prefactor of Z–R relation and b is the power of Z–R relation. Substituting $\left(\frac{6+\mu}{3+\gamma+\mu}\right)=b$, the equation (18) yields,

$$Z=\left[\frac{10^4}{6\pi c}\right]^{b}\left(\frac{\Gamma(\mu)}{\Gamma(3+\gamma+\mu)}\right)^{b}\frac{\Gamma(6+\mu)}{\Gamma(\mu)}\times$$

$$\times\left[N_T\left(\frac{3+\mu}{D_m}\right)^{\mu}\right]^{\frac{\gamma-3}{3+\gamma+\mu}}\left(\frac{3+\mu}{D_m}\right)^{-\frac{3\mu}{3+\gamma+\mu}} R^{b} \qquad (20)$$

Plot of b–μ is shown in figure1 for $\gamma = 0.67$. From the figure, it is seen that the graph gives an asymptotic nature for $\mu \le -3.67$. Simultaneously for $b=1$, the Z–R relationship in equation (20) becomes independent of N_T and for, $b=1.634$, the relationship is independent of D_m. So, two special cases arise, where the former may be regarded as the drop number independent case and the later one as the mean drop diameter independent case. Thus the range of b may be written as $1 \le b \le 1.634$, and hence, the values of $\mu \le -3.67$ are invalid. It is in good agreement with the value of μ provided by many of the earlier investigators (Ulbrich, 1983; Wills and Tottleman, 1989; Tokay and Short, 1996). But, analytically no upper limit of μ may be defined. However, it is found that for $\mu \le 0$, the integral parameters give negative moments which are not physically relevant. So the lower limit may be rescaled as $\mu \le 0$. Ulbrich and Atlas (1998) found that the central 60% data lies in the range $-1.5 \le \mu \le 3$ with three-parameter gamma distribution. Thus the scale of μ has the range $0 \le \mu \le 3$.

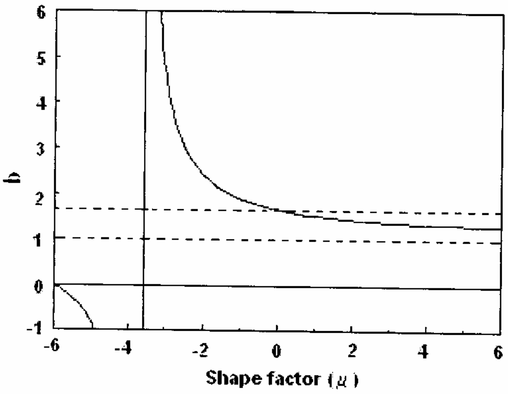

Fig. 1. Dependence of b on DSD shape parameter considering exponent of (3) as 0.67. b is the power of (Z-R) relation.

Rain Parameter Diagrams

Combining equations (9) and (12), one can get

$$N_0=\Lambda^{\mu}N_T=(3+\mu)^{\mu}\frac{N_T}{D_m^{\mu}} \qquad (21)$$

Equation (21) relates the intercept coefficient (N_0) with the number density of drops in a volume of air (N_T) and mass-weighted mean drop diameter (D_m). Simultaneously, it is dependent in the shape parameter μ. The dependence of N_T–D_m relationship on μ is shown in figure 2, for fixed N_0 (8000 $m^{-3}.mm^{-(1+\mu)}$) and R (10 $mm.h^{-1}$). Now considering μ=1.5, the plots of $N_T \perp D_m$ are given in figure 3 for different values of N_0 using (21), and also for different rain rates R using (15). Clearly figure 2 and figure 3 can be used for the selection of the Z–R relation to be used.

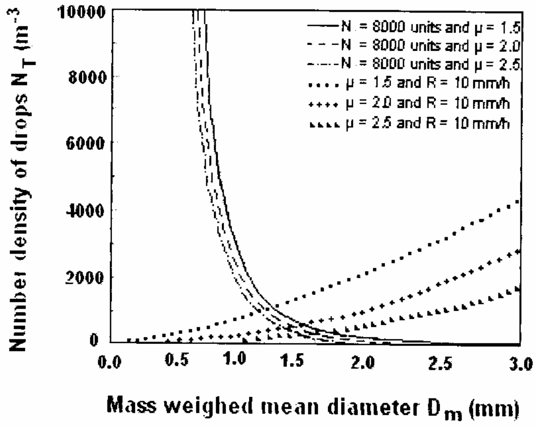

Fig. 2. N_T - D_m for different values of μ for eq. (21) and eq. (15).

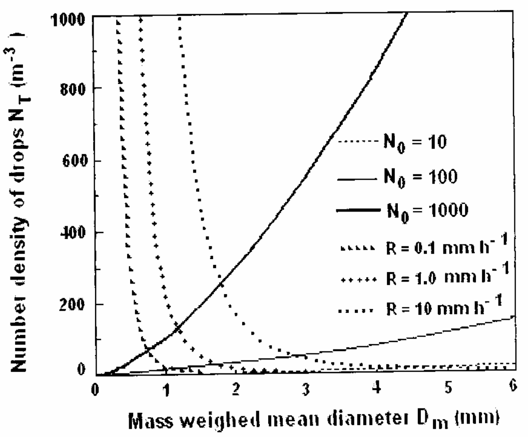

Fig. 3. N_T - D_m for different N_0 and R at constant μ according to eq. (21) and eq. (15) respectively.

Therefore, combining equations (15) and (21), the relationship between D_m and R can be established as

$$D_m = \left[\frac{10^4}{6\pi c} \frac{1}{N_0} \left\{ \frac{\Gamma(3+\gamma+\mu)}{\Gamma(\mu)} \right\} \right]^{\frac{1}{\mu(3+\gamma)}} \times$$

$$\times (3+\mu)^{\frac{1}{\mu}} R^{\frac{1}{\mu(3+\gamma)}} \quad (22)$$

The prefactor of the above equation depends on the shape parameter μ, the range of which is predicted before. So, considering the mean value of shape parameter for certain value of N_0 (8000 m^{-3}.mm$^{-(1+\mu)}$) and with Ulbrich's values of γ and c, one can get

$D_m = 1.886\, R^{0.18}$, (23)

the plot of which is shown in Fig.4 along with the results of Feingold and Levin (1986).

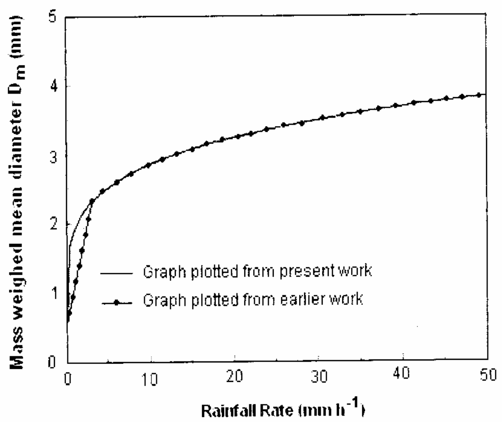

Fig. 4. Comparison of empirical relation and experimental results.

Therefore, using (12) in equation (22), the expression of slope coefficient can be obtained as

$$\Lambda = \left[\frac{6\pi c}{10^4} N_0 \frac{\Gamma(\mu)}{\Gamma(3+\gamma+\mu)} \right]^{\frac{1}{\mu(1+\gamma)}} \times$$

$$\times (3+\mu)^{\frac{\mu-1}{\mu}} . R^{-\frac{1}{\mu(3+\gamma)}} \quad (24)$$

In equation (24), the prefactor and the exponent relationship is solely dependent on the shape parameter μ. Considering the mean value of μ, Atlas and Ulbrich's values of γ and c and Marshall and Palmer's value of N_0, the equation (24) yields

$\Lambda = 3.708\, R^{-0.18}$ (25)

This Λ–R relationship differs a little from the proposed A–R relationship of Uijlenhoet (2001). Corresponding plot of Λ–R is shown in Fig.5.

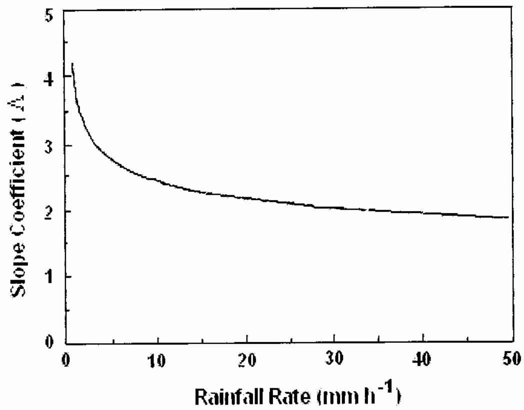

Fig. 5. Λ – R plot.

Combining equations (15) and (21), N_T–R relationship may be deduced, where the prefactor and the exponent are dependent on the shape parameter (μ) through

$$N_T = \left[\frac{10^4}{6\pi c} N_0^{\frac{3+\gamma}{\mu}} \frac{\Gamma(\mu)}{\Gamma(3+\gamma+\mu)} \right]^{\frac{\mu}{(3+\gamma+\mu)}} R^{\frac{\mu}{(3+\gamma+\mu)}} \quad (26)$$

For predicted range of μ and taking Atlas and Ulbrich's values of γ and c, and also Marshall and Palmer's value of N_0, the equation (26) yields

$N_T = 563.67\, R^{0.29}$ (27)

This N_T – R relationship is comparable to the results of Feingold and Levin (1986). The N_T –R plot has been shown in Fig.6.

CONCLUSION

This alternative approach for the determination of three parameters of DSD seems to be quite useful in the

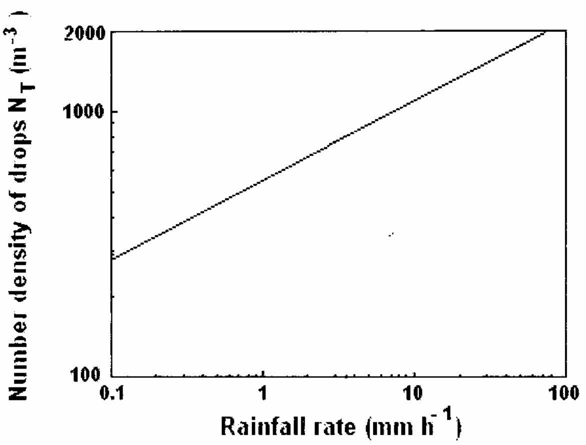

Fig. 6. N_T – R plot.

analyses of their relative variations under different circumstances. The method of approach can justify the results obtained from lognormal distribution. More refinement of this present analysis can be achieved through the inclusion of other processes, e.g., coalescence and evaporation in the estimation of raindrop size distribution. The interesting feature in this three parameter model lies in the selection of scaling parameter within the choice of empirical approximation for the estimation of drop size distribution. The other parametric variations derived from the present analysis have been compared with the experimental results of the earlier workers (Feingold and Levin, 1986; Uijlenhoet, 2001).

The drop size distribution which is represented hereby the general form of gamma distribution is found to have better correspondence with experimental results (Maiciel and Assis, 1990). The prefactors of *Z–R* relations are dependent on the type of rainfall (convective or stratiform). So, under different climatological conditions, *Z–R* relation may vary. Also, from figure, it is seen that D_m increases very rapidly for the stratiform rain, where as, for the convective types of rain, D_m increases very slowly. In fact, for higher rain rates ($R > 40$ min.h^{-1}), μ becomes almost constant. The slow variation of D_m suggests that DSD might be approaching to the steady state condition.

The graphical method of solution for different values of N_T / D_m has been determined in this work which may be taken as a process to resolve the uncertainty in determining the variation of *Z* with *R*.

ACKNOWLEDGEMENT

This work has been funded by Indian Space Research Organization (ISRO) through S.K. Mitra Centre for Research in Space Environment, University of Calcutta, Kolkata, India.

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